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Difference equations for correlation functions of $A^{(1)}_{n-1}$-face model with boundary reflection

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Abstract

The $A^{(1)}_{n-1}$-face model with boundary reflection is considered on the basis of the boundary corner transfer matrix bootstrap. We construct the fused boundary Boltzmann weights to determine the normalization factor. We derive difference equations of the quantum Knizhnik–Zamolodchikov type for correlation functions of the boundary model. The simplest difference equations are solved in the case of the free boundary condition.

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1. Introduction

Many two-dimensional integrable models without boundaries have been solved by using the representation theory of affine quantum groups [1, 2]. The integrability of bulk models is ensured by the factorized scattering condition or the Yang–Baxter equation, in addition to the unitarity and crossing symmetry condition [3, 4].

Cherednik [5, 6] showed that the integrability in the presence of a reflecting boundary is ensured by the reflection equation (boundary Yang–Baxter equation) and the Yang–Baxter equation for bulk theory. A systematic treatment for determining the spectrum of integrable models with boundary reflection was initiated by Sklyanin [7] in the framework of the algebraic Bethe ansatz. The boundary interaction is specified by the boundary S-matrix for massive quantum theories [8], by the reflection matrix for lattice vertex models [7], and by the boundary weights for lattice face models [9, 10].

In our previous paper [11] Belavin's $\mathbb{Z}_n$-symmetric elliptic vertex model with boundary reflection is considered on the basis of the boundary CTM (corner transfer matrix) bootstrap formulated in [12]. We derived a set of difference equations, called the boundary quantum Knizhnik–Zamolodchikov equations, for correlation functions in the boundary Belavin model. Furthermore, we obtained the boundary spontaneous polarization by solving the simplest difference equations. The resulting quantities are exactly equal to the square of that for bulk spontaneous polarization [13], up to a phase factor.

In this paper we consider the $A^{(1)}_{n-1}$-face model [14] with a boundary on the basis of a boundary CTM bootstrap. The $\mathbb{Z}_n$-symmetric model and the $A^{(1)}_{n-1}$-face model are related by the vertex-face correspondence [14] in bulk theory. Thus we wish to find the similar structure in the boundary $A^{(1)}_{n-1}$-face model as that observed in [11].

Integrable face models on a half infinite lattice have been studied in [15–18]. In [15] the transfer matrix of the boundary ABF model (the boundary $A^{(1)}_1$-face model) was diagonalized by constructing the boundary vacuum states, using the ansatz [19] that the boundary vacuum states should be obtained from the Fock vacuum states by applying the exponential of the infinite sum of the quadratic bosonic oscillators associated with the bulk ABF models [20]. In [16] the solution to the reflection equation was given for the boundary $A^{(1)}_{n-1}$-face model. In [17] free energy and critical exponents were obtained for $n = 2$, the boundary ABF model case. In [18] correlation functions for the boundary XYZ model were obtained in terms of those for the boundary ABF model [15] by using the vertex-face transformation method [21].

It was shown in [8] that the boundary K-matrix can be determined, up to a CDD factor, by imposing the reflection equation (boundary Yang–Baxter equation), the boundary unitarity and the boundary crossing symmetry. In [16] the boundary crossing symmetries were derived for the boundary face models associated with $A^{(1)}_1$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$ and $A^{(2)}_n$, but not for the model associated with $A^{(1)}_{n-1}$ ($n > 2$). In order to discuss the higher n cases we shall establish the boundary crossing symmetry (3.20) for $A^{(1)}_{n-1}$, without which the K-matrix cannot be exactly determined. Our main goal in this paper is to derive difference equations (4.12) satisfied by correlation functions.

The rest of this paper is organized as follows. In section 2 we review the boundary $A^{(1)}_{n-1}$-face model, thereby fixing our notation. In section 3 we introduce the fusion of the K-matrix to determine the normalization factors of the K-matrix. We also establish the boundary crossing symmetry. In section 4 we construct a lattice realization of the vertex operators on the basis of the boundary CTM bootstrap approach. Furthermore, we derive difference equations for correlation functions of the boundary $A^{(1)}_{n-1}$-face model. In section 5 we give some concluding remarks. In appendix A we give the explicit expressions of fused Boltzmann weights of the bulk $A^{(1)}_{n-1}$-face model [22]. In appendix B we give a simple sketch of the proof of the reflection equation.

2. Boundary $A^{(1)}_{n-1}$-face model

The present section aims to formulate the problem, thereby fixing the notation.

2.1. Theta functions

Throughout this paper we fix the integers n and r such that $r \geqslant n+2$, and also fix the parameter x such that $0 < x < 1$. We will use the abbreviations

$$[z] = x^{(z^2/r)-z}\Theta_{x^{2r}}(x^{2z}) \tag{2.1}$$

where the Jacobi theta function is given by

$$\Theta_q(\zeta) = (\zeta; q)_\infty (q\zeta^{-1}; q)_\infty (q, q)_\infty \tag{2.2}$$

$$(\zeta; q_1, \ldots, q_m) = \prod_{i_1,\ldots,i_m \geqslant 0} (1 - \zeta q_1^{i_1} \cdots q_m^{i_m}). \tag{2.3}$$

For an additive parameter z, we often use the corresponding multiplicative parameter $\zeta = x^{2z}$.

For later convenience we also introduce the following symbols:

$$r_m(z) = \zeta^{-[(r-1)/r][(n-m)/n]} \frac{g_m(\zeta)}{g_m(\zeta^{-1})} \qquad g_m(\zeta) = \frac{\{x^{2n+2r-m-1}\zeta\}\{x^{m+1}\zeta\}}{\{x^{2n-m+1}\zeta\}\{x^{2r+m-1}\zeta\}} \tag{2.4}$$

where $1 \leqslant m \leqslant n-1$ and

$$\{\zeta\} = (\zeta; x^{2n}, x^{2r})_\infty. \tag{2.5}$$

In particular the function $r_1(z)$ will appear in the expression for the Boltzmann weights of the $A_{n-1}^{(1)}$-face model in regime III.

2.2. *The weight lattice of $A_{n-1}^{(1)}$*

Let $V = \mathbb{C}^n$ and $\{\varepsilon_\mu\}_{0 \leqslant \mu \leqslant n-1}$ be the standard orthogonal basis with the inner product $\langle \varepsilon_\mu, \varepsilon_\nu \rangle = \delta_{\mu\nu}$. The weight lattice of $A_{n-1}^{(1)}$ is defined as follows:

$$P = \bigoplus_{\mu=0}^{n-1} \mathbb{Z}\bar{\varepsilon}_\mu \tag{2.6}$$

where

$$\bar{\varepsilon}_\mu = \varepsilon_\mu - \varepsilon \qquad \varepsilon = \frac{1}{n}\sum_{\mu=0}^{n-1} \varepsilon_\mu.$$

We denote the fundamental weights by ω_μ $(1 \leqslant \mu \leqslant n-1)$:

$$\omega_\mu = \bar{\varepsilon}_0 + \bar{\varepsilon}_1 + \cdots + \bar{\varepsilon}_{\mu-1}.$$

Since $\omega_n = 0$, you can define ω_μ for $\mu \in \mathbb{Z}$ by setting $\omega_{\mu+n} = \omega_\mu$. For $a \in P$ we set

$$a_{\mu\nu} = \bar{a}_\mu - \bar{a}_\nu \qquad \bar{a}_\mu = \langle a + \rho, \varepsilon_\mu \rangle \qquad \rho = \sum_{\mu=1}^{n-1} \omega_\mu. \tag{2.7}$$

We also set

$$P_l^+ = \left\{ \sum_{\mu=1}^{n-1} a^\mu \omega_\mu \,\middle|\, a^1, \ldots, a^{n-1} \in \mathbb{Z}_{\geqslant 0}, \sum_{\mu=1}^{n-1} a^\mu \leqslant l \right\}.$$

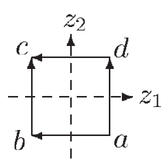
We may denote $a \in P_l^+$ by $a = \sum_{\mu=1}^{n-1} a^\mu \omega_\mu + \left(l - \sum_{\mu=1}^{n-1} a^\mu\right)\omega_0$.

2.3. *The $A_{n-1}^{(1)}$-face model*

The $A_{n-1}^{(1)}$-face model is the one whose local state a is restricted such that $a \in P_{r-n}^+$. An ordered pair $(a, b) \in P^2$ is called *admissible* if $b = a + \bar{\varepsilon}_\mu$, for a certain μ $(0 \leqslant \mu \leqslant n-1)$. In what follows we denote $b \longleftarrow a$ when $(a, b) \in P^2$ is admissible. Furthermore, an ordered quartet $(a, b, c, d) \in P^4$ is called *admissible* if the four pairs (a, b), (a, d), (b, c) and (d, c) are admissible.

For $(a, b, c, d) \in P^4$ let

$$W(a, b, c, d|z_1 - z_2) = W\left(\begin{matrix} c & d \\ b & a \end{matrix} \,\middle|\, z_1 - z_2\right) =$$



be the Boltzmann weight of the $A^{(1)}_{n-1}$ model for the state configuration (a, b, c, d) round a face. Here the four states a, b, c and d are ordered clockwise from the SE corner, and the oriented broken lines in the above figure carry spectral parameters. In this model $W(a, b, c, d|z) = 0$ unless the quartet (a, b, c, d) is admissible. Non-zero Boltzmann weights are parametrized in terms of the elliptic theta function of the spectral parameter z as follows:

$$\begin{aligned}
&W\left(\begin{matrix} a+2\bar{\varepsilon}_\mu & a+\bar{\varepsilon}_\mu \\ a+\bar{\varepsilon}_\mu & a \end{matrix}\middle| z\right) = r_1(z) \\
&W\left(\begin{matrix} a+\bar{\varepsilon}_\mu+\bar{\varepsilon}_\nu & a+\bar{\varepsilon}_\mu \\ a+\bar{\varepsilon}_\nu & a \end{matrix}\middle| z\right) = r_1(z)\frac{[z][a_{\mu\nu}-1]}{[z+1][a_{\mu\nu}]} \qquad (\mu \neq \nu) \\
&W\left(\begin{matrix} a+\bar{\varepsilon}_\mu+\bar{\varepsilon}_\nu & a+\bar{\varepsilon}_\nu \\ a+\bar{\varepsilon}_\nu & a \end{matrix}\middle| z\right) = r_1(z)\frac{[z+a_{\mu\nu}][1]}{[z+1][a_{\mu\nu}]} \qquad (\mu \neq \nu)
\end{aligned} \tag{2.8}$$

where $-\frac{n}{2} < z < 0$ in regime III.

The Boltzmann weights (2.8) solve the face-type Yang–Baxter equation [14]:

$$\begin{aligned}
&\sum_g W\left(\begin{matrix} d & e \\ c & g \end{matrix}\middle| z_1\right) W\left(\begin{matrix} c & g \\ b & a \end{matrix}\middle| z_2\right) W\left(\begin{matrix} e & f \\ g & a \end{matrix}\middle| z_1 - z_2\right) \\
&\qquad = \sum_g W\left(\begin{matrix} g & f \\ b & a \end{matrix}\middle| z_1\right) W\left(\begin{matrix} d & e \\ g & f \end{matrix}\middle| z_2\right) W\left(\begin{matrix} d & g \\ c & b \end{matrix}\middle| z_1 - z_2\right).
\end{aligned} \tag{2.9}$$

Some numerical calculations concerning the hard hexagon model in [4] suggest that the CTM is well defined in the thermodynamic limit if the normalization factor $r_1(z)$ is chosen such that the partition function per site is equal to unity. In order to fix $r_1(z)$ the following two inversion relations are useful [14]:

$$\sum_g W\left(\begin{matrix} c & g \\ b & a \end{matrix}\middle| -z\right) W\left(\begin{matrix} c & d \\ g & a \end{matrix}\middle| z\right) = \delta_{bd} \tag{2.10}$$

$$\sum_g G_g W\left(\begin{matrix} g & b \\ d & c \end{matrix}\middle| -n-z\right) W\left(\begin{matrix} g & d \\ b & a \end{matrix}\middle| z\right) = \delta_{ac}\frac{G_b G_d}{G_a} \tag{2.11}$$

where

$$G_a = \prod_{0\leqslant\mu<\nu\leqslant n-1} [a_{\mu\nu}].$$

From the inversion trick based on these relations we get the expression of $r_1(z)$ in regime III.

The Boltzmann weights (2.8) also have σ-invariance [14]:

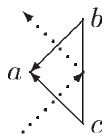
$$W\left(\begin{matrix} \sigma(c) & \sigma(d) \\ \sigma(b) & \sigma(a) \end{matrix}\middle| z\right) = W\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| z\right) \tag{2.12}$$

where σ is the diagram automorphism of $A^{(1)}_{n-1}$ defined by $\sigma(\omega_\mu) = \omega_{\mu+1}$.

2.4. Solution to the reflection equation

Let us consider the interaction at the boundary, which is specified by the boundary Boltzmann weight or the K-matrix:

$$K(a, b, c|z) = K\left(a \begin{matrix} b \\ c \end{matrix}\middle| z\right) =$$



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Here (b, a) and (c, a) are admissible. The integrability condition of the boundary face model is the face-type reflection equation [9, 10]:

$$\sum_{d,e} K\left(f\,\begin{matrix}g\\e\end{matrix}\middle|z_2\right) W\left(\begin{matrix}c & f\\ d & e\end{matrix}\middle|z_1+z_2\right) K\left(d\,\begin{matrix}e\\a\end{matrix}\middle|z_1\right) W\left(\begin{matrix}c & d\\ b & a\end{matrix}\middle|z_1-z_2\right)$$
$$= \sum_{d,e} W\left(\begin{matrix}c & f\\ d & g\end{matrix}\middle|z_1-z_2\right) K\left(d\,\begin{matrix}g\\e\end{matrix}\middle|z_1\right) W\left(\begin{matrix}c & d\\ b & e\end{matrix}\middle|z_1+z_2\right) K\left(b\,\begin{matrix}e\\a\end{matrix}\middle|z_2\right). \tag{2.13}$$

For the bulk Boltzmann weights of the $A^{(1)}_{n-1}$-face model [14] the diagonal solution to equation (2.13) is given as follows [16]:

$$K\left(a+\bar{\varepsilon}_\mu\,\begin{matrix}a\\a'\end{matrix}\middle|z\right) = f_a(z)\frac{[\bar{a}_\mu+\eta+z]}{[\bar{a}_\mu+\eta-z]}\delta_{aa'} \tag{2.14}$$

where $\eta = \eta(a)$ may depend on a but is a constant with respect to z. In the present paper we take an a-independent constant η for simplicity. The authors of [15] employed the diagonal K-matrix for the boundary ABF model ($n = 2$ case) with $\eta(a) = k/2 + c$ for $a = (k-1)\omega_1 + (r-1-k)\omega_0$. We also notice that the opposite admissible conditions, such that (a, b) and (a, c) are admissible, were used in [16]. Thus we give a simple sketch of the proof of the claim that (2.14) solves the reflection equation (2.13) in appendix B.

Let $a \in P^+_{r-n}$ be the local state. Then in regime III any ground state configuration is labelled by some $b \in P^+_{r-n-1}$, i.e. a configuration consisting of the cyclic sequence of $b, b+\omega_1, \ldots, b+\omega_{n-1}$ [14]. In the 'low-temperature' limit $x \to 0$, one of these ground states is realized. In what follows we fix one of them (say, labelled by b) and define all the correlation functions in terms of the 'low-temperature' series expansion, the formal power series with respect to x. Then the fixed ground state configuration gives the lowest order. Furthermore, any finite order contribution results from the configurations which differ from that of the fixed ground state by altering a finite number of local states. Thus the infinite number of states at far enough sites should coincide with the fixed ground state configuration labelled by b. Such one-to-one correspondence with the ground state configuration allows us to specify the boundary conditions by the same index $b \in P^+_{r-n-1}$.

When we fix the local state $a \in P^+_{r-n}$ at the right-most corner in the bulk case, there are n possible ground state configurations labelled by $b = a - \omega_i \in P^+_{r-n-1}$ $(0 \leqslant i \leqslant n-1)$. In the presence of the boundary weight (2.14), the σ invariance (2.12) is broken and one of $b = a - \omega_i \in P^+_{r-n-1}$ is selected as the ground state configuration when we fix $a \in P^+_{r-n}$ at the right-most corner. In order to determine which $b = a - \omega_i$ labels the ground state configuration, i.e. which $K(a+\bar{\varepsilon}_i, a, a|z)$ takes the largest among the $K(a+\bar{\varepsilon}_\mu, a, a|z)$, let us consider the boundary Boltzmann weights in the 'low-temperature' limit $x \to 0$.

Here we assume that the constant η belongs to one of the following n disjoint intervals:

$$\left(-\frac{\bar{a}_0+\bar{a}_1}{2}, -\bar{a}_1\right), \left(-\frac{\bar{a}_1+\bar{a}_2}{2}, -\bar{a}_2\right), \ldots, \left(-\frac{\bar{a}_{n-2}+\bar{a}_{n-1}}{2}, -\bar{a}_{n-1}\right),$$
$$\left(-\bar{a}_{n-1}, \frac{r-\bar{a}_0-\bar{a}_{n-1}}{2}\right).$$

Note that

$$\bar{a}_{n-1} < \cdots < \bar{a}_1 < \bar{a}_0 < r + \bar{a}_{n-1} \qquad \bar{a}_{\mu-1} - \bar{a}_\mu \in \mathbb{Z}_{>0}\ (1 \leqslant \mu \leqslant n-1) \tag{2.15}$$

for $a \in P^+_{r-n}$ [23]. We further restrict the spectral parameter z to satisfy

$$-\frac{n}{2} < -\bar{a}_{n-1} - \eta < z < 0 \qquad \text{if} \quad -\bar{a}_{n-1} < \eta < \frac{r-\bar{a}_0-\bar{a}_{n-1}}{2} \tag{2.16}$$

$$-\frac{n}{2} < \bar{a}_i + \eta < z < 0 \qquad \text{if} \quad -\frac{\bar{a}_{i-1} + \bar{a}_i}{2} < \eta < -\bar{a}_i \qquad (1 \leqslant i \leqslant n-1). \tag{2.17}$$

When $-\bar{a}_{n-1} < \eta < \frac{r-\bar{a}_0-\bar{a}_{n-1}}{2}$, by using (2.15) and (2.16) we have

$$\bar{a}_\mu + \eta - z > \bar{a}_\mu + \eta + z \geqslant \bar{a}_{n-1} + \eta + z > 0 \qquad (0 \leqslant \mu \leqslant n-1)$$
$$r - \bar{a}_\mu - \eta - z > r - \bar{a}_\mu - \eta + z \geqslant r - \bar{a}_0 - \eta + z > 0 \qquad (0 \leqslant \mu \leqslant n-1).$$

Thus the boundary Boltzmann weights behave like

$$f_a(z)^{-1} K\left(a + \bar{\varepsilon}_\mu \, \begin{matrix} a \\ a' \end{matrix} \middle| z \right) \sim \zeta^{[2(\bar{a}_\mu+\eta)/r]-1} \qquad (x \to 0). \tag{2.18}$$

We therefore find from (2.18) that $K(a+\bar{\varepsilon}_0, a, a|z)$ takes the largest for $-\bar{a}_{n-1} < \eta < \frac{r-\bar{a}_0-\bar{a}_{n-1}}{2}$.

When $-\frac{\bar{a}_{i-1}+\bar{a}_i}{2} < \eta < -\bar{a}_i$ $(1 \leqslant i \leqslant n-1)$, by using (2.15) and (2.17) we have

$$\bar{a}_\mu + \eta - z > \bar{a}_\mu + \eta + z \geqslant \bar{a}_{i-1} + \eta + z > 0 \qquad (0 \leqslant \mu \leqslant i-1)$$
$$\bar{a}_\mu + \eta + z < \bar{a}_\mu + \eta - z \leqslant \bar{a}_i + \eta - z < 0 \qquad (i \leqslant \mu \leqslant n-1)$$
$$r - \bar{a}_\mu - \eta - z > r - \bar{a}_\mu - \eta + z \geqslant r - \bar{a}_0 - \eta + z > 0 \qquad (0 \leqslant \mu \leqslant n-1).$$

Thus the boundary Boltzmann weights behave like

$$f_a(z)^{-1} K\left(a + \bar{\varepsilon}_\mu \, \begin{matrix} a \\ a' \end{matrix} \middle| z \right) \sim \begin{cases} \zeta^{[2(\bar{a}_\mu+\eta)/r]-1} & (0 \leqslant \mu \leqslant i-1) \\ \zeta^{[2(\bar{a}_\mu+\eta)/r]+1} & (i \leqslant \mu \leqslant n-1) \end{cases} \qquad (x \to 0). \tag{2.19}$$

We therefore find from (2.19) that $K(a+\bar{\varepsilon}_i, a, a|z)$ takes the largest value for $-\frac{\bar{a}_{i-1}+\bar{a}_i}{2} < \eta < -\bar{a}_i$ $(1 \leqslant i \leqslant n-1)$.

As is seen above, which $K(a+\bar{\varepsilon}_i, a, a|z)$ takes the largest value depends on the value of η. Suppose that we fix η such that $K(a+\bar{\varepsilon}_i, a, a|z)$ takes the largest value among the $K(a+\bar{\varepsilon}_\mu, a, a|z)$. Then the boundary condition can be labelled by

$$b = a - \omega_i \qquad (0 \leqslant i \leqslant n-1). \tag{2.20}$$

For fixed i in (2.20) we should rather rewrite (2.14) as the expression normalized by $K(a+\bar{\varepsilon}_i, a, a|z)$:

$$K^{(i)}\left(a + \bar{\varepsilon}_\mu \, \begin{matrix} a \\ a' \end{matrix} \middle| z \right) = f_a^{(i)}(z) \frac{[\bar{a}_i + \eta - z]}{[\bar{a}_i + \eta + z]} \frac{[\bar{a}_\mu + \eta + z]}{[\bar{a}_\mu + \eta - z]} \delta_{aa'}. \tag{2.21}$$

3. Fusion of K-matrices

In order to determine the normalization factor $f_a^{(i)}(z)$ let us introduce the fusion of K-matrices. The fusion hierarchy of the boundary ABF model (boundary $A_1^{(1)}$-face model) was constructed in [10], and the fusion procedure of the boundary vertex models was considered in [24, 25].

3.1. Bulk and boundary face operators

In this subsection we reformulate the bulk Boltzmann weight W and the boundary Boltzmann weight K as elements of the bulk and boundary face operators. Let

$$\Omega_z^{(b,a)} = \begin{cases} \mathbb{C}v^{(b,a)} & \text{if} \quad (a,b) \in P^2 \text{ is admissible} \\ 0 & \text{otherwise} \end{cases}$$
$$\Omega_z = \bigoplus_{a,b} \Omega_z^{(b,a)}.$$

Then the W operator is defined as [23]

$$W^{\Omega_{z_1},\Omega_{z_2}}(v^{(d,a)}\otimes v^{(c,d)}) = \sum_b v^{(c,b)}\otimes v^{(b,a)} W\left(\begin{matrix} c & d \\ b & a\end{matrix}\middle| z_1 - z_2\right). \tag{3.1}$$

Furthermore, if we introduce the K operator by

$$\begin{aligned} &K(z) : \Omega_z \to \Omega_{-z} \\ &K(z)v^{(b,a)} = \sum_{a'} v^{(b,a')} K\left(b \begin{matrix} a \\ a'\end{matrix}\middle| z\right)\end{aligned} \tag{3.2}$$

the reflection equation (2.13) can be regarded as the equality of linear operators:

$$K_2(z_2)W_{21}^{\Omega_{z_2},\Omega_{-z_1}}K_1(z_1)W_{12}^{\Omega_{z_1},\Omega_{z_2}} = W_{21}^{\Omega_{-z_2},\Omega_{-z_1}}K_1(z_1)W_{12}^{\Omega_{z_1},\Omega_{-z_2}}K_2(z_2) \tag{3.3}$$

where the subscripts to the W and K denote the spaces on which they nontrivially act. Then both sides of (3.3) map $\Omega_{z_1}\otimes\Omega_{z_2}$ to $\Omega_{-z_1}\otimes\Omega_{-z_2}$.

3.2. *Fusion procedure of the W operator*

For fixed m $(2\leqslant m\leqslant n)$ let $z_j = z+\frac{m+1}{2}-j$ $(1\leqslant j\leqslant m)$, and for $0\leqslant \mu_1 < \cdots < \mu_m \leqslant n-1$ we denote

$$a_0 = a \qquad a_j = a + \bar{\varepsilon}_{\mu_1} + \cdots + \bar{\varepsilon}_{\mu_j}\ (1\leqslant j\leqslant m)$$

and

$$a_j^\sigma = a + \bar{\varepsilon}_{\mu_{\sigma(1)}} + \cdots + \bar{\varepsilon}_{\mu_{\sigma(j)}}\ (1\leqslant j\leqslant m)$$

for $\sigma\in\mathfrak{S}_m$. Note that $a_m^\sigma = a_m$. Let

$$\begin{aligned} &\wedge^m(\Omega_z^{(a_m,a_0)}) = \sum_{\sigma\in\mathfrak{S}_m}(\operatorname{sgn}\sigma)\,\Omega_{z_1}^{(a_1^\sigma,a_0)}\otimes\Omega_{z_2}^{(a_2^\sigma,a_1^\sigma)}\otimes\cdots\otimes\Omega_{z_m}^{(a_m,a_{m-1}^\sigma)} \\ &\wedge^m(\Omega_z) = \bigoplus_{a_0,a_m}\wedge^m(\Omega_z^{(a_m,a_0)})\end{aligned}$$

and let $v^{(a_m,a)}$ stand for the one-dimensional basis of $\wedge^m(\Omega_z^{(a_m,a_0)})$. Then $W_{21}^{\Omega_{z_2},\Omega_{z_1}}$ is the fusion operator associated with $\wedge^2(\Omega_z)$, because $\operatorname{Im}(W_{21}^{\Omega_{z_2},\Omega_{z_1}}|\Omega_{z_1}\otimes\Omega_{z_2}) = \wedge^2(\Omega_z)$. For general m, the fusion operators $\pi_\pm^{(m)}$ associated with the $\wedge^m(\Omega_{\pm z})$ are given as follows [23]:

$$\begin{aligned} \pi_+^{(m)} &= W_{m\,m-1}^{\Omega_{z_m},\Omega_{z_{m-1}}}\cdots W_{32}^{\Omega_{z_3},\Omega_{z_2}}W_{31}^{\Omega_{z_3},\Omega_{z_1}}W_{21}^{\Omega_{z_2},\Omega_{z_1}} \\ \pi_-^{(m)} &= W_{12}^{\Omega_{-z_1},\Omega_{-z_2}}W_{13}^{\Omega_{-z_1},\Omega_{-z_3}}W_{23}^{\Omega_{-z_2},\Omega_{-z_3}}\cdots W_{m-1\,m}^{\Omega_{-z_{m-1}},\Omega_{-z_m}}.\end{aligned}$$

The m-fold fused W operator as an intertwiner on $\Omega_{z_1}\otimes\wedge^m(\Omega_{z_2})$ should be defined as

$$W^{\Omega_{z_1},\wedge^m(\Omega_{z_2})}(v^{(d,a)}\otimes v^{(d_m,d)}) = \sum_{a_m} v^{(d_m,a_m)}\otimes v^{(a_m,a)}W^{(1,m)}\left(\begin{matrix} d_m & d \\ a_m & a\end{matrix}\middle| z_1 - z_2\right) \tag{3.4}$$

where the $W^{(1,m)}$ are the horizontal m-fold fused Boltzmann weights, and $d_m - d = \bar{\varepsilon}_{\lambda_1}+\cdots+\bar{\varepsilon}_{\lambda_m}$ with $0\leqslant\lambda_1<\cdots<\lambda_m\leqslant n-1$. Another m-fold fused W operator as an intertwiner on $\wedge^m(\Omega_{z_1})\otimes\Omega_{z_2}$ should be defined as

$$W^{\wedge^m(\Omega_{z_1}),\Omega_{z_2}}(v^{(a_m,a)}\otimes v^{(b_m,a_m)}) = \sum_b v^{(b_m,b)}\otimes v^{(b,a)}W^{(m,1)}\left(\begin{matrix} b_m & a_m \\ b & a\end{matrix}\middle| z_1 - z_2\right) \tag{3.5}$$

where the $W^{(m,1)}$ are the vertical m-fold fused Boltzmann weights, and $b_m - b = \bar{\varepsilon}_{\lambda_1}+\cdots+\bar{\varepsilon}_{\lambda_m}$ with $0\leqslant\lambda_1<\cdots<\lambda_m\leqslant n-1$. See [22] for the definitions of horizontal and vertical fused Boltzmann weights. The results are also summarized in appendix A of this paper.

Furthermore, we denote the dual space of Ω_z by $\Omega_z^* \cong \wedge^{n-1}(\Omega_z)$. Let

$$\Omega_z^{*(b,a)} = \begin{cases} \mathbb{C}v^{*(b,a)} & \text{if} \quad (b,a) \in P^2 \text{ is admissible} \\ 0 & \text{otherwise} \end{cases}$$

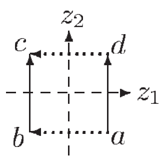
$$\Omega_z^* = \bigoplus_{a,b} \Omega_z^{(b,a)}.$$

The dual W operators are defined as

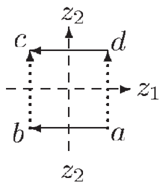
$$\begin{aligned} W^{\Omega_{z_1},\Omega^*_{z_2}}(v^{(d,a)} \otimes v^{*(c,d)}) &= \sum_b v^{(c,b)} \otimes v^{*(b,a)} W^{\Omega_{z_1},\Omega^*_{z_2}} \begin{pmatrix} c & d \\ b & a \end{pmatrix} \\ W^{\Omega^*_{z_1},\Omega_{z_2}}(v^{*(d,a)} \otimes v^{(c,d)}) &= \sum_b v^{*(c,b)} \otimes v^{(b,a)} W^{\Omega^*_{z_1},\Omega_{z_2}} \begin{pmatrix} c & d \\ b & a \end{pmatrix} \\ W^{\Omega^*_{z_1},\Omega^*_{z_2}}(v^{*(d,a)} \otimes v^{*(c,d)}) &= \sum_b v^{*(c,b)} \otimes v^{*(b,a)} W^{\Omega^*_{z_1},\Omega^*_{z_2}} \begin{pmatrix} c & d \\ b & a \end{pmatrix}. \end{aligned} \tag{3.6}$$

The dual Boltzmann weights are graphically represented as follows:

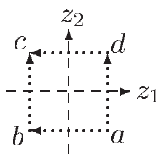
$$W^{\Omega_{z_1},\Omega^*_{z_2}} \begin{pmatrix} c & d \\ b & a \end{pmatrix} =$$



$$W^{\Omega^*_{z_1},\Omega_{z_2}} \begin{pmatrix} c & d \\ b & a \end{pmatrix} =$$



$$W^{\Omega^*_{z_1},\Omega^*_{z_2}} \begin{pmatrix} c & d \\ b & a \end{pmatrix} =$$



Here, $b \leftarrow\cdots\cdots a$ implies that $(b,a) \in P^2$ is admissible.

3.3. Fusion procedure of the K operator

Now we wish to construct the m-fold fusion of K-operator mapping $\wedge^m(\Omega_z)$ to $\wedge^m(\Omega_{-z})$:

$$K^{(m)}(z)v^{(a_m,a)} = \sum_{a'} v^{(a_m,a')} K^{(m)} \left(a_m \begin{matrix} a \\ a' \end{matrix} \middle| z \right). \tag{3.7}$$

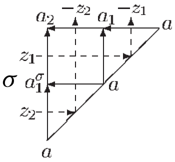
Here we use the same notation z_j, a_j, a_j^σ $(1 \leqslant j \leqslant m)$ as in the previous subsection.

When $m = 2$ the fused K-matrix is defined as follows:

$$\begin{aligned} K_+^{(2)} \left(a_2 \begin{matrix} a \\ a' \end{matrix} \middle| z \right) &= \sum_{\sigma \in \mathfrak{S}_2} \operatorname{sgn}\sigma \sum_{a''} K_1 \left(a_1 \begin{matrix} a \\ a'' \end{matrix} \middle| z_1 \right) \\ &\quad \times W_{12} \left(\begin{matrix} a_2 & a_1 \\ a_1^\sigma & a'' \end{matrix} \middle| z_1 + z_2 \right) K_2 \left(a_1^\sigma \begin{matrix} a'' \\ a' \end{matrix} \middle| z_2 \right) \\ &= \delta_{aa'} \sum_{\sigma \in \mathfrak{S}_2} \operatorname{sgn}\sigma\, K_1 \left(a_1 \begin{matrix} a \\ a \end{matrix} \middle| z_1 \right) W_{12} \left(\begin{matrix} a_2 & a_1 \\ a_1^\sigma & a \end{matrix} \middle| z_1 + z_2 \right) K_2 \left(a_1^\sigma \begin{matrix} a \\ a \end{matrix} \middle| z_2 \right). \end{aligned} \tag{3.8}$$

In the last equality we use the diagonal property of the K-matrix, i.e. $K(a,b,c|z)=0$ unless $b\neq c$. The two-fold K-matrix is represented graphically as follows:

$$K_{+}^{(2)}\left(a_2\,\begin{matrix}a\\a'\end{matrix}\,\middle|\,z\right)=\sum_{\sigma\in\mathfrak{S}_2}\operatorname{sgn}\sigma$$



.

For general $m\geqslant 2$ and $\sigma\in\mathfrak{S}_m$, let us introduce the symbols $a^{(k,j)}$ $(0\leqslant j\leqslant k\leqslant m)$ such that

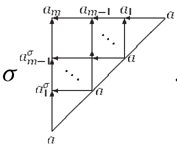
$$\begin{array}{lll} a^{(0,0)}=a & a^{(k,0)}=a_k & (1\leqslant k\leqslant m)\\ a^{(m,m)}=a' & a^{(m,j)}=a^{\sigma}_{m-j} & (1\leqslant j\leqslant m-1). \end{array}$$

In this notation, the summation variable a'' in (3.8) is denoted by $a^{(1,1)}$ with $m=2$. Using these symbols the m-fold fusion of K-matrices is defined as follows:

$$\begin{aligned} K_{+}^{(m)}\left(a_m\,\begin{matrix}a\\a'\end{matrix}\,\middle|\,z\right)&=\sum_{\sigma\in\mathfrak{S}_m}\operatorname{sgn}\sigma\sum_{\{a^{(k,j)}\}}\prod_{j=1}^{m}K\left(a^{(j,j-1)}\,\begin{matrix}a^{(j-1,j-1)}\\a^{(j,j)}\end{matrix}\,\middle|\,z_j\right)\\ &\quad\times\prod_{\substack{j,k=1\\ j\leqslant k}}^{m-1}W\left(\begin{matrix}a^{(k,j-1)} & a^{(k-1,j-1)}\\ a^{(k,j)} & a^{(k-1,j)}\end{matrix}\,\middle|\,z_j+z_k\right)=\delta_{aa'}K_{+}^{(m)}\left(a_m\,\begin{matrix}a\\a\end{matrix}\,\middle|\,z\right). \end{aligned}\tag{3.9}$$

In the last equality we again use the diagonal property of the K-matrix. The m-fold K-matrix is represented graphically as follows:

$$K_{+}^{(m)}\left(a_m\,\begin{matrix}a\\a\end{matrix}\,\middle|\,z\right)=\sum_{\sigma\in\mathfrak{S}_m}\operatorname{sgn}\sigma$$



.

It is evident that the m-fold K-matrix can be constructed in an inductive manner:

$$\begin{aligned} K_{+}^{(m)}\left(a_m\,\begin{matrix}a\\a\end{matrix}\,\middle|\,z\right)&=\sum_{a_1^{\sigma}=a+\bar{\varepsilon}_{\mu_{\sigma(1)}}}(-1)^{\sigma(1)-1}K_{+}^{(m-1)}\left(a_{m-1}\,\begin{matrix}a\\a\end{matrix}\,\middle|\,z+\tfrac{1}{2}\right)\\ &\quad\times W^{(m-1,1)}\left(\begin{matrix}a & a_{m-1}\\ a_1^{\sigma} & a\end{matrix}\,\middle|\,z+\tfrac{1}{2}+z_m\right)K\left(a_1^{\sigma}\,\begin{matrix}a\\a\end{matrix}\,\middle|\,z_m\right)\\ &=\sum_{a_{m-1}^{\sigma}=a_m-\bar{\varepsilon}_{\mu_{\sigma(m)}}}(-1)^{m-\sigma(m)}K\left(a_1\,\begin{matrix}a\\a\end{matrix}\,\middle|\,z_1\right)\\ &\quad\times W^{(1,m-1)}\left(\begin{matrix}a & a_1\\ a_{m-1}^{\sigma} & a\end{matrix}\,\middle|\,z-\tfrac{1}{2}+z_1\right)K_{+}^{(m-1)}\left(a_{m-1}^{\sigma}\,\begin{matrix}a\\a\end{matrix}\,\middle|\,z-\tfrac{1}{2}\right). \end{aligned}\tag{3.10}$$

Here $W^{(1,m-1)}$ and $W^{(m-1,1)}$ are the horizontal and vertical fused Boltzmann weights, respectively.

Successive application of the Yang–Baxter equation (2.9) and the reflection equation (2.13) indicate the reflection equations involving $K_{+}^{(m)}(z)$:

$$\begin{aligned} &K_2(z_2)W_{21}^{\Omega_{z_2},\wedge^m(\Omega_{-z_1})}K_{+1}^{(m)}(z_1)W_{12}^{\wedge^m(\Omega_{z_1}),\Omega_{z_2}}\\ &\qquad=W_{21}^{\Omega_{-z_2},\wedge^m(\Omega_{-z_1})}K_{+1}^{(m)}(z_1)W_{12}^{\wedge^m(\Omega_{z_1}),\Omega_{-z_2}}K_2(z_2)\\ &K_{+2}^{(m)}(z_2)W_{21}^{\wedge^m(\Omega_{z_2}),\wedge^m(\Omega_{-z_1})}K_{+1}^{(m)}(z_1)W_{12}^{\wedge^m(\Omega_{z_1}),\wedge^m(\Omega_{z_2})}\\ &\qquad=W_{21}^{\wedge^m(\Omega_{-z_2}),\wedge^m(\Omega_{-z_1})}K_{+1}^{(m)}(z_1)W_{12}^{\wedge^m(\Omega_{z_1}),\wedge^m(\Omega_{-z_2})}K_{+2}^{(m)}(z_2). \end{aligned}\tag{3.11}$$

Both sides of the first of equations (3.11) map $\wedge^m(\Omega_{z_1}) \otimes \Omega_{z_2}$ to $\wedge^m(\Omega_{-z_1}) \otimes \Omega_{-z_2}$, while those of the second of equations (3.11) map $\wedge^m(\Omega_{z_1}) \otimes \wedge^m(\Omega_{z_2})$ to $\wedge^m(\Omega_{-z_1}) \otimes \wedge^m(\Omega_{-z_2})$.

Another fused K-matrix is defined in an inductive manner as follows. For $m = 2$ let

$$K_-^{(2)}\left(a_2 \begin{matrix} a \\ a \end{matrix} \middle| z\right) = \sum_{\sigma \in \mathfrak{S}_2} \operatorname{sgn}\sigma\, K_2\left(a_1 \begin{matrix} a \\ a \end{matrix} \middle| z_2\right) W_{21}\left(\begin{matrix} a_2 & a_1 \\ a_1^\sigma & a \end{matrix} \middle| z_1 + z_2\right) K_1\left(a_1^\sigma \begin{matrix} a \\ a \end{matrix} \middle| z_1\right) \tag{3.12}$$

and for $m > 2$ let

$$\begin{aligned} K_-^{(m)}\left(a_m \begin{matrix} a \\ a \end{matrix} \middle| z\right) &= \sum_{a_1^\sigma = a + \bar{\varepsilon}_{\mu_{\sigma(1)}}} (-1)^{\sigma(1)-1} K_+^{(m-1)}\left(a_{m-1} \begin{matrix} a \\ a \end{matrix} \middle| z - \tfrac{1}{2}\right) \\ &\quad \times W^{(m-1,1)}\left(\begin{matrix} a & a_{m-1} \\ a_1^\sigma & a \end{matrix} \middle| z - \tfrac{1}{2} + z_1\right) K\left(a_1^\sigma \begin{matrix} a \\ a \end{matrix} \middle| z_1\right) \\ &= \sum_{a_{m-1}^\sigma = a_m - \bar{\varepsilon}_{\mu_{\sigma(m)}}} (-1)^{m-\sigma(m)} K\left(a_1 \begin{matrix} a \\ a \end{matrix} \middle| z_m\right) \\ &\quad \times W^{(1,m-1)}\left(\begin{matrix} a & a_1 \\ a_{m-1}^\sigma & a \end{matrix} \middle| z + \tfrac{1}{2} + z_m\right) K_+^{(m-1)}\left(a_{m-1}^\sigma \begin{matrix} a \\ a \end{matrix} \middle| z + \tfrac{1}{2}\right). \end{aligned} \tag{3.13}$$

Two kinds of m-fold fused K-matrices are related as follows:

$$K_+^{(m)}(z)\pi_+^{(m)} = \pi_-^{(m)} K_-^{(m)}(z). \tag{3.14}$$

Note that (3.14) with $m = 2$ directly follows from the reflection equation (2.13), and also that (3.14) with $m > 2$ follows from the Yang–Baxter equation (2.9) and the reflection equation (2.13). The relation (3.14) implies that $\operatorname{Im}(K_+^{(m)}(z)| \wedge^m (\Omega_z)) = \wedge^m(\Omega_{-z})$, and hence that $K_+^{(m)}(z)$ can be regarded as a bona fide K-matrix mapping $\wedge^m(\Omega_z)$ to $\wedge^m(\Omega_{-z})$ satisfying (3.11).

3.4. Boundary crossing symmetry

By taking account of the recursion relation (3.10), the explicit expression of $K_+^{(m)}(z)$ can be obtained as follows:

$$\begin{aligned} K_+^{(m)}\left(a + \sum_{j=1}^m \bar{\varepsilon}_{\mu_j} \begin{matrix} a \\ a \end{matrix} \middle| z\right) &= (-1)^{{}_mC_2} \prod_{j<k} r_1(z_j + z_k) \prod_{j=1}^{m-1} \frac{[2z - j]}{[2z + j]} \\ &\quad \times \prod_{j=1}^m f_a^{(i)}(z_j) \frac{[\bar{a}_i + \eta - z_j]}{[\bar{a}_i + \eta + z_j]} \frac{[\bar{a}_{\mu_j} + \eta + z_1]}{[\bar{a}_{\mu_j} + \eta - z_n]}. \end{aligned} \tag{3.15}$$

Note that the sign factor $(-1)^{{}_mC_2}$ results from the permutation $(1, \ldots, m) \mapsto (m, \ldots, 1)$. See the graphical representation of $K_+^{(m)}$ following equation (3.9).

Since $\wedge^n(\Omega_z^{(a,a)}) \cong \mathbb{C}$, the n-fold fused K-matrix should be a scalar. By putting the scalar equal to unity we obtain the normalization factor $f_a^{(i)}(z)$ of the K-matrix in (2.21) as follows:

$$f_a^{(i)}(z) = \zeta^{[(n-1)/n][(r-1)/r]-2\bar{a}_i/r} \frac{g(\zeta)p_a^{(i)}(\zeta)p_a^{(i)}(x^2\zeta^{-1})}{g(\zeta^{-1})p_a^{(i)}(\zeta^{-1})p_a^{(i)}(x^2\zeta)} \tag{3.16}$$

where $\zeta = x^{2z}$, and

$$g(\zeta) = \frac{(x^{2n+2}\zeta^2; x^{4n}, x^{2r})_\infty (x^{2(r+n-1)}\zeta^2; x^{4n}, x^{2r})_\infty}{(x^{2r}\zeta^2; x^{4n}, x^{2r})_\infty (x^{4n}\zeta^2; x^{4n}, x^{2r})_\infty}$$

$$p_a^{(i)}(\zeta) = \prod_{j=0}^{n-1} \frac{(x^{2(\bar{a}_i+\eta+j)}\zeta; x^{2n}, x^{2r})_\infty}{(x^{2(r+n-j-1-\bar{a}_i-\eta)}\zeta; x^{2n}, x^{2r})_\infty} \frac{(x^{2(r-\bar{a}_j-\eta)}\zeta; x^{2n}, x^{2r})_\infty}{(x^{2(\bar{a}_j+\eta+n-1)}\zeta; x^{2n}, x^{2r})_\infty}.$$

Let us define the dual K operator and its matrix element as follows:

$$K^*(z)v^{*(b,a)} = \sum_{a'} v^{*(b,a')} K^* \left(b \begin{matrix} a \\ a' \end{matrix} \middle| z \right). \tag{3.17}$$

When $m = n-1$ we identify the dual K-matrix with the fused K-matrix as follows:

$$K^* \left(a - \bar{\varepsilon}_\mu \begin{matrix} a \\ a' \end{matrix} \middle| z \right) = K_+^{(n-1)} \left(a + \sum_{\substack{\nu=0 \\ \nu\neq\mu}}^{n-1} \bar{\varepsilon}_\nu \begin{matrix} a \\ a' \end{matrix} \middle| z \right). \tag{3.18}$$

For later convenience we also introduce the $\hat{K}$ operator:

$$\hat{K}(z)v^{(b,a)} = \sum_{b'} v^{(b',a)} K^* \left(a \begin{matrix} b \\ b' \end{matrix} \middle| -z - \frac{n}{2} \right). \tag{3.19}$$

The boundary crossing symmetry can be obtained by substituting (3.18) into (3.10) with $m = n$:

$$K^* \left(b \begin{matrix} a \\ a' \end{matrix} \middle| z \right) = \delta_{aa'} \sum_d \frac{G_d}{G_a} W \left(\begin{matrix} d & a \\ a & b \end{matrix} \middle| 2z \right) K \left(d \begin{matrix} a \\ a \end{matrix} \middle| -\frac{n}{2} - z \right). \tag{3.20}$$

The boundary crossing relations were found in [7, 25] for vertex-type models, and in [10, 16] for face-type models. As for the $A^{(1)}_1$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$ and $A^{(2)}_n$-face model cases, see [16]. This is the first to derive the boundary crossing relation for the boundary $A^{(1)}_{n-1}$-face model with $n > 2$. The vertex-type version of (3.20) is given by (4.12) in [11].

4. Correlation functions and difference equations

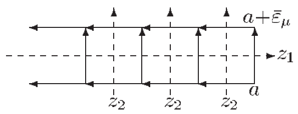
4.1. Vertex operators and commutation relations

For $b = a - \omega_i$ $(0 \leqslant i \leqslant n-1)$, let $\mathcal{H}_{l,k}$ be the space of admissible paths $(\ldots, a_2, a_1, a_0)$ such that

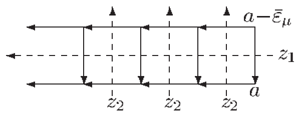
$$\begin{aligned} &a_0 = a \qquad a_j - a_{j-1} \in \{\bar{\varepsilon}_0, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_{n-1}\} \qquad \text{for} \quad j = 1, 2, 3, \ldots \\ &a_j = b + \omega_{j+i} \qquad \text{for} \quad j \gg 1 \end{aligned} \tag{4.1}$$

where $k = a + \rho$, $l = b + \rho$.

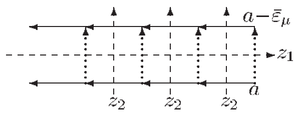
Following [2, 22] we can identify the type I vertex operators with the half transfer matrices. Here we need four types of vertex operators:



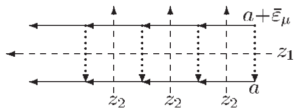
$$= \phi_b^{(a+\bar{\varepsilon}_\mu,a)}(z_1 - z_2) : \mathcal{H}_{l,k} \longrightarrow \mathcal{H}_{l,k+\bar{\varepsilon}_\mu}$$



$$= \phi^b_{(a-\bar{\varepsilon}_\mu,a)}(z_2 - z_1) : \mathcal{H}_{l,k} \longrightarrow \mathcal{H}_{l,k-\bar{\varepsilon}_\mu}$$



$$= \phi_b^{*(a-\bar{\varepsilon}_\mu,a)}(z_1 - z_2) : \mathcal{H}_{l,k} \longrightarrow \mathcal{H}_{l,k-\bar{\varepsilon}_\mu}$$



$$= \phi^{*b}_{(a+\bar{\varepsilon}_\mu,a)}(z_2 - z_1) : \mathcal{H}_{l,k} \longrightarrow \mathcal{H}_{l,k+\bar{\varepsilon}_\mu}.$$

In what follows we often suppress the letter b specifying the boundary condition.

It follows from the Yang–Baxter equation (2.9) and the boundary condition that these vertex operators satisfy the following commutation relations:

$$\begin{aligned}
\phi^{(c,d)}(z_2)\phi^{(d,a)}(z_1) &= \sum_d W\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| z_1 - z_2\right)\phi^{(c,d)}(z_1)\phi^{(d,a)}(z_2) \\
\phi^{*(c,b)}(z_2)\phi^{(b,a)}(z_1) &= \sum_d W^{\Omega_{z_1},\Omega^*_{z_2}}\begin{pmatrix} c & d \\ b & a \end{pmatrix}\phi^{(c,d)}(z_1)\phi^{*(d,a)}(z_2) \\
\phi^{(c,b)}(z_2)\phi^{*(b,a)}(z_1) &= \sum_d W^{\Omega^*_{z_1},\Omega_{z_2}}\begin{pmatrix} c & d \\ b & a \end{pmatrix}\phi^{*(c,d)}(z_1)\phi^{(d,a)}(z_2) \\
\phi^{*(c,b)}(z_2)\phi^{*(b,a)}(z_1) &= \sum_d W^{\Omega^*_{z_1},\Omega^*_{z_2}}\begin{pmatrix} c & d \\ b & a \end{pmatrix}\phi^{*(c,d)}(z_1)\phi^{*(d,a)}(z_2).
\end{aligned} \tag{4.2}$$

Here $W^{\Omega_{z_1},\Omega^*_{z_2}}$, $W^{\Omega^*_{z_1},\Omega_{z_2}}$ and $W^{\Omega^*_{z_1},\Omega^*_{z_2}}$ denote the dual W operators. See appendix A concerning their matrix elements.

Furtermore, the unitarity relations with respect to the W operators imply the inversion relation of the vertex operators:

$$\sum_{\mu=0}^{n-1}\phi_{(a,a+\bar{\varepsilon}_\mu)}(-z)\phi^{(a+\bar{\varepsilon}_\mu,a)}(z) = 1. \tag{4.3}$$

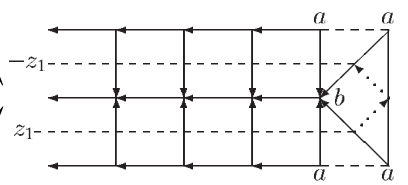
Thanks to the crossing symmetry with respect to the W operators (see appendix A) we also have the duality identities:

$$\phi^{*(a-\bar{\varepsilon}_\mu,a)}(z) = G_a^{-1}\phi_{(a-\bar{\varepsilon}_\mu,a)}\left(-z-\frac{n}{2}\right) \qquad \phi^*_{(a+\bar{\varepsilon}_\mu,a)}(z) = G_a\phi^{(a+\bar{\varepsilon}_\mu,a)}\left(-z-\frac{n}{2}\right). \tag{4.4}$$

Using the vertex operators introduced in the previous section, the transfer matrix for the semi-infinite lattice is expressed as follows:

$$T_B^{(i)}(z) = \sum_{\mu=0}^{n-1}\phi_{(a,a+\bar{\varepsilon}_\mu)}(z)K^{(i)}\left(a+\bar{\varepsilon}_\mu\begin{matrix} a \\ a\end{matrix}\middle| z\right)\phi^{(a+\bar{\varepsilon}_\mu,a)}(z)$$

$$= \sum_b \tag{4.5}$$



From (4.5) and (4.4) we also have another expression:

$$T_B^{(i)}(z) = \sum_{\mu=0}^{n-1}G_{a+\bar{\varepsilon}_\mu}\phi^{*(a,a+\bar{\varepsilon}_\mu)}(z)K^{(i)}\left(a+\bar{\varepsilon}_\mu\begin{matrix} a \\ a\end{matrix}\middle| z\right)\phi^{(a+\bar{\varepsilon}_\mu,a)}(z). \tag{4.6}$$

4.2. Derivation of difference equations

In sections 2 and 3 we fix the normalization of W and K such that the maximal eigenvalues of the boundary transfer matrix $T_B^{(i)}(z)$ are equal to unity in the thermodynamic limit. Thus the boundary vacuum state $|k-\omega_i, k\rangle_B$ in $\mathcal{H}_{k-\omega_i,k}$ and its dual ${}_B\langle k-\omega_i, k|$ in $\mathcal{H}^*_{k-\omega_i,k}$ should satisfy

$$\begin{aligned}
&K^{(i)}\left(a+\bar{\varepsilon}_\mu\begin{matrix} a \\ a\end{matrix}\middle| z\right)\phi^{(a+\bar{\varepsilon}_\mu,a)}(z)|k-\omega_i,k\rangle_B = \phi^{(a+\bar{\varepsilon}_\mu,a)}(-z)|k-\omega_i,k\rangle_B \\
&{}_B\langle k-\omega_i,k|\phi^*_{(a+\bar{\varepsilon}_\mu,a)}(z)K^{(i)}\left(a+\bar{\varepsilon}_\mu\begin{matrix} a \\ a\end{matrix}\middle| z\right) = {}_B\langle k-\omega_i,k|\phi^*_{(a+\bar{\varepsilon}_\mu,a)}(-z).
\end{aligned} \tag{4.7}$$

Define the $(N+1)$-point correlation function as

$$F^{(i)}(z_1, z_2, \ldots, z_N)^{a_0,a_1,a_2,\ldots,a_{N-1},a_N} = {}_B\langle k-\omega_i, k|\phi^{(a_0,a_1)}(z_1)\phi^{(a_1,a_2)}(z_2)\cdots\phi^{(a_{N-1},a_N)}(z_N)|k-\omega_i, k\rangle_B \tag{4.8}$$

where we assume that $N \equiv 0 \bmod n$ for simplicity. It follows from (4.2), (4.7) and (4.4) that correlation functions should satisfy

(1) R-matrix symmery:

$$F^{(i)}(\ldots, z_{j+1}, z_j, \ldots)^{\ldots,a_{j-1},a_j,a_{j+1}\ldots} = \sum_{a'_j} W\left(\begin{matrix} a_{j+1} & a_j \\ a'_j & a_{j-1}\end{matrix}\middle| z_1 - z_2\right) F^{(i)}(\ldots, z_j, z_{j+1}, \ldots)^{\ldots,a_{j-1},a'_j,a_{j+1}\ldots} \tag{4.9}$$

(2) reflection properties:

$$F^{(i)}(\ldots, -z_N)^{\ldots,a_{N-1},a_N} = K^{(i)}\left(a_{N-1}\,\begin{matrix} a_N \\ a_N\end{matrix}\middle| z\right) F^{(i)}(\ldots, z_N)^{\ldots,a_{N-1},a_N} \tag{4.10}$$

$$F^{(i)}\left(-z_1 - \frac{n}{2}, \ldots,\right)^{a_0,a_1,\ldots} = \hat{K}^{(i)}\left(a_1\,\begin{matrix} a_0 \\ a_0\end{matrix}\middle| z\right) F^{(i)}(z_1, \ldots)^{a_0,a_1,\ldots}. \tag{4.11}$$

These relations can be shown by the same discussion as in [11, 12].

Let

$$F^{(i)}(z_1, z_2, \ldots, z_N) = \sum_{a_0,a_1,\ldots,a_N} v^{(a_0,a_1)} \otimes v^{(a_1,a_2)} \otimes \cdots \otimes v^{(a_{N-1},a_N)} \times F^{(i)}(z_1, z_2, \ldots, z_N)^{a_0,a_1,a_2,\ldots,a_{N-1},a_N}.$$

Then we conclude from (4.9)–(4.11) that the $\Omega_{z_1} \otimes \cdots \otimes \Omega_{z_N}$-valued correlation function $F^{(i)}(z_1, \ldots, z_N)$ should satisfy the following difference equations:

$$\begin{aligned} T_j F^{(i)}_N(z_1, \ldots, z_N) &= W^{\Omega_{z_j-n},\Omega_{z_{j-1}}}_{jj-1} \cdots W^{\Omega_{z_j-n},\Omega_{z_1}}_{j1} \hat{K}_j(-z_j) \\ &\times W^{\Omega_{z_1},\Omega_{-z_j}}_{1j} \cdots W^{\Omega_{z_{j-1}},\Omega_{-z_j}}_{j-1j} W^{\Omega_{z_{j+1}},\Omega_{-z_j}}_{j+1j} \cdots W^{\Omega_{z_N},\Omega_{-z_j}}_{Nj} \\ &\times K_j(z_j) W^{\Omega_{z_j},\Omega_{z_N}}_{jN} \cdots W^{\Omega_{z_j},\Omega_{z_{j+1}}}_{jj+1} F^{(i)}_N(z_1, \ldots, z_N) \end{aligned} \tag{4.12}$$

where

$$T_j f(z_1, \ldots, z_j, \ldots, z_N) = f(z_1, \ldots, z_j - n, \ldots, z_N).$$

4.3. Simple difference equations

In this subsection we consider the correlation functions of the form

$$\begin{aligned} P^{(a+\bar{\varepsilon}_\mu,a)}_i(z_1, z_2) &= {}_B\langle k-\omega_i, k|\phi_{(a,a+\bar{\varepsilon}_\mu)}(-z_1)\phi^{(a+\bar{\varepsilon}_\mu,a)}(z_2)|k-\omega_i, k\rangle_B \\ &= G_{a+\bar{\varepsilon}_\mu} \times {}_B\langle k-\omega_i, k|\phi^{*(a,a+\bar{\varepsilon}_\mu)}(-z_1)\phi^{(a+\bar{\varepsilon}_\mu,a)}(z_2)|k-\omega_i, k\rangle_B. \end{aligned} \tag{4.13}$$

You can show in a similar way to (4.12) that these correlation functions satisfy the following difference equations:

$$\begin{aligned} T_1 P^{(a+\bar{\varepsilon}_\lambda,a)}_i(z_1, z_2) &= \sum_{\mu\nu} \frac{G_{a+\bar{\varepsilon}_\lambda} G_{a-\bar{\varepsilon}_\mu}}{G_a^2} K^{(i)}\left(a+\bar{\varepsilon}_\lambda\,\begin{matrix} a \\ a\end{matrix}\middle| z_1 - n\right) \\ &\times W\left(\begin{matrix} a+\bar{\varepsilon}_\lambda & a \\ a & a-\bar{\varepsilon}_\mu\end{matrix}\middle| -z_1 - z_2\right) K^{*(i)}\left(a-\bar{\varepsilon}_\mu\,\begin{matrix} a \\ a\end{matrix}\middle| z_1 - \frac{n}{2}\right) \\ &\times W\left(\begin{matrix} a+\bar{\varepsilon}_\nu & a \\ a & a-\bar{\varepsilon}_\mu\end{matrix}\middle| z_2 - z_1\right) P^{(a+\bar{\varepsilon}_\nu,a)}_i(z_1, z_2) \end{aligned} \tag{4.14}$$

$$T_2 P_i^{(a+\bar{\varepsilon}_\lambda,a)}(z_1,z_2) = \sum_{\mu\nu} \frac{G_{a+\bar{\varepsilon}_\lambda} G_{a-\bar{\varepsilon}_\mu}}{G_a^2} W\left(\begin{matrix} a+\bar{\varepsilon}_\lambda & a \\ a & a-\bar{\varepsilon}_\mu \end{matrix}\middle| z_1 - z_2\right)$$
$$\times K^{*(i)}\left(a-\bar{\varepsilon}_\mu \begin{matrix} a \\ a \end{matrix}\middle| z_2 - \frac{n}{2}\right) W\left(\begin{matrix} a+\bar{\varepsilon}_\nu & a \\ a & a-\bar{\varepsilon}_\mu \end{matrix}\middle| -z_1 - z_2\right)$$
$$\times K^{*(i)}\left(a+\bar{\varepsilon}_\nu \begin{matrix} a \\ a \end{matrix}\middle| z_1 - n\right) P_i^{(a+\bar{\varepsilon}_\nu,a)}(z_1,z_2). \tag{4.15}$$

Set

$$P_i^{(a)}(z_1,z_2) = \sum_{\lambda=0}^{n-1} P_i^{(a+\bar{\varepsilon}_\lambda,a)}(z_1,z_2).$$

Then we restrict ourselves to the limiting case such that a K-matrix is a certain scalar, as done in [11,12]. Then the difference equations for $P_i^{(a)}(z_1,z_2)$ can be derived from (4.14) and (4.15) as follows:

$$\begin{aligned} \frac{T_1 P_i^{(a)}(z_1,z_2)}{P_i^{(a)}(z_1,z_2)} &= (x^{-2n}\zeta_1^2)^{(n-1)/r} r_1(2z_1-n) r_1(-z_+) \frac{[-z_+ + n]}{[-z_+ + 1]} r_1(-z_-) \frac{[-z_- + n]}{[-z_- + 1]} \\ \frac{T_2 P_i^{(a)}(z_1,z_2)}{P_i^{(a)}(z_1,z_2)} &= (x^{-2n}\zeta_2^2)^{(n-1)/r} r_1(2z_2-n) r_1(-z_+) \frac{[-z_+ + n]}{[-z_+ + 1]} r_1(z_-) \frac{[z_- + n]}{[z_- + 1]}. \end{aligned} \tag{4.16}$$

Here we use the notation $z_\pm = z_1 \pm z_2$, and also use the following sum formulae:

$$\begin{aligned} \sum_{\lambda=0}^{n-1} \frac{G_{a+\bar{\varepsilon}_\lambda}}{G_a} W\left(\begin{matrix} a+\bar{\varepsilon}_\lambda & a \\ a & a-\bar{\varepsilon}_\mu \end{matrix}\middle| z\right) &= r_1(z) \frac{[z+n]}{[z+1]} \\ \sum_{\mu=0}^{n-1} \frac{G_{a-\bar{\varepsilon}_\mu}}{G_a} W\left(\begin{matrix} a+\bar{\varepsilon}_\nu & a \\ a & a-\bar{\varepsilon}_\mu \end{matrix}\middle| z\right) &= r_1(z) \frac{[z+n]}{[z+1]}. \end{aligned} \tag{4.17}$$

The solution to (4.16) is given as follows:

$$P_i^{(a)}(z_1,z_2) = C_i^{(a)} A(z_1) A(z_2) B(z_+) B(z_-) \tag{4.18}$$

where $C_i^{(a)}$ is a constant, and

$$A(z) = a(\zeta^2) a(\zeta^{-2}) \qquad a(\zeta) = \frac{(x^{4n+2r-2}\zeta; x^{2n}, x^{4n}, x^{2r})_\infty (x^{2n+2}\zeta; x^{2n}, x^{4n}, x^{2r})_\infty}{(x^{2n+2r}\zeta; x^{2n}, x^{4n}, x^{2r})_\infty (x^{4n}\zeta; x^{2n}, x^{4n}, x^{2r})_\infty}$$
$$B(z) = b(\zeta) b(\zeta^{-1}) \qquad b(\zeta) = \frac{(x^{2r}\zeta; x^{2n}, x^{2n}, x^{2r})_\infty (x^{4n}\zeta; x^{2n}, x^{2n}, x^{2r})_\infty}{(x^{2n+2r-2}\zeta; x^{2n}, x^{2n}, x^{2r})_\infty (x^{2n+2}\zeta; x^{2n}, x^{2n}, x^{2r})_\infty}.$$

Before finishing this section let us prove the sum formulae (4.17), both of which are equivalent to

$$\prod_{\substack{\nu=0 \\ \nu\neq\mu}}^{n-1} \frac{[a_{\mu\nu}+1]}{[a_{\mu\nu}]} + \sum_{\substack{\lambda=0 \\ \lambda\neq\mu}}^{n-1} \frac{[z+a_{\lambda\mu}+1][1]}{[z+1][a_{\lambda\mu}]} \prod_{\substack{\nu=0 \\ \nu\neq\lambda,\mu}}^{n-1} \frac{[a_{\lambda\nu}+1]}{[a_{\lambda\nu}]} = \frac{[z+n]}{[z+1]}. \tag{4.19}$$

Note that the LHS of (4.19) is not singular at $\bar{a}_i = \bar{a}_j$ $(0 \leqslant i < j \leqslant n-1)$, and also that the LHS vanishes at $z = -n$. Hence the LHS is equal to the RHS up to a constant C. In order to show that $C = 1$, it is sufficient to set $z = 0$ and formally set

$$a_{\mu\nu} = \begin{cases} n+\nu-\mu & (\nu = 0, 1, \ldots, \mu-1) \\ \nu-\mu & (\nu = \mu+1, \ldots, n-1). \end{cases}$$

Thus the sum formulae (4.17) are proved.

5. Concluding remarks

Let us briefly summarize the results of this paper. In section 2 we discussed the ground state configurations for the boundary $A_{n-1}^{(1)}$-face model. Here we restricted the spectral parameter z to satisfy one of n conditions (2.16) and (2.17). Accordingly, one of n ground state configurations is specified. At this stage the K-matrix was determined up to a scalar factor $f_a^{(i)}(z)$ (see (2.21)). We determined the factor $f_a^{(i)}(z)$ by imposing the boundary crossing symmetry (3.20). For that purpose the fusion procedure of K operators was established in section 3. Using the type I vertex operators [2, 22] we constructed the $\Omega_{z_1} \otimes \cdots \otimes \Omega_{z_N}$-valued correlation functions that should satisfy the difference equation (4.12). We solved the simplest difference equations to obtain the one-point local state probability (4.18).

There are two ways of proceeding further. One is solving the difference equations (4.12) to obtain the corresponding local state probabilities, while the other is constructing the boundary vacuum states in terms of bosonized vertex operators to do the same thing. Let us discuss the latter way here.

Consider the bosons B_m^j $(1 \leqslant j \leqslant n-1, m \in \mathbb{Z}\backslash\{0\})$ with the commutation relations [26, 27]

$$[B_m^j, B_{m'}^k] = \begin{cases} m\dfrac{[(n-1)m]_x}{[nm]_x}\dfrac{[(r-1)m]_x}{[rm]_x}\delta_{m+m',0} & (j=k) \\ -mx^{\mathrm{sgn}(j-k)nm}\dfrac{[m]_x}{[nm]_x}\dfrac{[(r-1)m]_x}{[rm]_x}\delta_{m+m',0} & (j\neq k) \end{cases}$$

where the symbol $[a]_x$ stands for $(x^a - x^{-a})/(x - x^{-1})$. Define B_m^n by

$$\sum_{j=1}^{n} x^{-2jm} B_m^j = 0.$$

Using these oscillators B_m^j the bosonization of the type I vertex operators for the $A_{n-1}^{(1)}$-face model was given in [22] on the Fock space $\mathcal{F}_{l,k}$. Furthermore, we make the ansatz [15, 19] such that the boundary vacuum states and their dual have the form

$$|k-\omega_i, k\rangle_B = \exp(F_a^{(i)})|k-\omega_i, k\rangle \qquad {}_B\langle k-\omega_i, k| = {}_B\langle k-\omega_i, k|\exp(G_a^{(i)})$$

where $|l, k\rangle$ is the highest weight of $\mathcal{F}_{l,k}$, and

$$F_a^{(i)} = -\tfrac{1}{2}\sum_{m>0}\sum_{s,t=1}^{n-1}\alpha_m^{st} B_{-m}^s B_{-m}^t + \sum_{m>0}\sum_{t=1}^{n-1}\beta_{m,a}^{t,(i)} B_{-m}^t$$

$$G_a^{(i)} = -\tfrac{1}{2}\sum_{m>0}\sum_{s,t=1}^{n-1}\gamma_m^{st} B_{-m}^s B_{-m}^t + \sum_{m>0}\sum_{t=1}^{n-1}\delta_{m,a}^{t,(i)} B_{-m}^t.$$

Since $\mathcal{H}_{l,k}$ and $\mathcal{F}_{l,k}$ have different characters, the bosonized expressions of correlation functions cannot be identified with the one defined in (4.8). In order to obtain the correct bosonized formulae for correlation functions, we have to construct the BRST cohomology of the appropriate complex which realizes the space of physical states $\mathcal{H}_{l,k}$ as subquotients of $\mathcal{F}_{l,k}$. We will address this problem in a separate paper.

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Appendix A. Fused $A^{(1)}_{n-1}$ Boltzmann weights

Let us introduce the fusion of Boltzmann weights W [22].

Let $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ be a subset of $N = \{0, 1, \ldots, n-1\}$ such that $\lambda_1 < \cdots < \lambda_m$. For $\kappa, \mu \in N$ set $\mu = \kappa$ if $\kappa \in \Lambda$, otherwise set $\mu \in \Lambda \cup \{\kappa\}$. For given κ, μ, Λ let $0 \leqslant \nu_1 < \cdots < \nu_m \leqslant n-1$ be such that $\bar{\varepsilon}_\mu + \bar{\varepsilon}_{\nu_1} + \cdots + \bar{\varepsilon}_{\nu_m} = \bar{\varepsilon}_\kappa + \bar{\varepsilon}_{\lambda_1} + \cdots + \bar{\varepsilon}_{\lambda_m}$.

The fusion of W in the horizontal direction is constructed as follows. Let $a, b, d = d_0, d_1, \ldots, d_{m-1}, d_m = c \in P$ satisfy

$$c = b + \bar{\varepsilon}_\mu \qquad d_j - d_{j-1} = \bar{\varepsilon}_{\lambda_j}\ (1 \leqslant j \leqslant m) \qquad d = a + \bar{\varepsilon}_\kappa.$$

Note that $b = a + \bar{\varepsilon}_{\nu_1} + \cdots + \bar{\varepsilon}_{\nu_m}$ from the definition of ν_j. Let $\sigma \in \mathfrak{S}_m$ be a permutation of $(1, \ldots, m)$, and set

$$a^\sigma_0 = a \qquad a^\sigma_j = b^\sigma_{j-1} + \bar{\varepsilon}_{\nu_{\sigma(j)}}\ (1 \leqslant j \leqslant m) \qquad a^\sigma_m = b.$$

Then m-fold antisymmetric fusion of W in the horizontal direction is given as

$$W^{(1,m)}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| z\right) = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \prod_{j=1}^{m} W\left(\begin{matrix} d_j & d_{j-1} \\ a^\sigma_j & a^\sigma_{j-1} \end{matrix}\middle| z + \frac{m+1}{2} - j\right). \tag{A.1}$$

Note that $W^{(1,m)}$ is antisymmetric with respect to $(\lambda_1, \ldots, \lambda_m)$.

Next, consider the fusion in the vertical direction. We use the same κ, μ, λ_j and ν_j as before. Now we set

$$b = a + \bar{\varepsilon}_\mu \qquad a_j - a_{j-1} = \bar{\varepsilon}_{\lambda_j}\ (1 \leqslant j \leqslant m) \qquad c = d + \bar{\varepsilon}_\kappa$$

where $a_0 = a$, $a_m = d$. We have $c = b + \bar{\varepsilon}_{\nu_1} + \cdots + \bar{\varepsilon}_{\nu_m}$. For $\sigma \in \mathfrak{S}_m$ set

$$b^\sigma_0 = b \qquad b^\sigma_j = b^\sigma_{j-1} + \bar{\varepsilon}_{\nu_{\sigma(j)}}\ (1 \leqslant j \leqslant m) \qquad b^\sigma_m = c.$$

Then m-fold antisymmetric fusion of W in the vertical direction is given as

$$W^{(m,1)}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| v\right) = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \prod_{j=1}^{m} W\left(\begin{matrix} b^\sigma_j & a_j \\ b^\sigma_{j-1} & a_{j-1} \end{matrix}\middle| z - \frac{m+1}{2} + j\right). \tag{A.2}$$

Note that $W^{(m,1)}$ is antisymmetric with respect to $(\lambda_1, \ldots, \lambda_m)$.

We further introduce the fusion of W in both the horizontal and vertical directions. Let $0 \leqslant \kappa_1 < \cdots < \kappa_m \leqslant n-1$, $0 \leqslant \mu_1 < \cdots < \mu_m \leqslant n-1$, $0 \leqslant \lambda_1 < \cdots < \lambda_{m'} \leqslant n-1$ and $0 \leqslant \nu_1 < \cdots < \nu_{m'} \leqslant n-1$ satisfy

$$\sum_{j=1}^{m} \bar{\varepsilon}_{\kappa_j} + \sum_{j=1}^{m'} \bar{\varepsilon}_{\lambda_j} = \sum_{j=1}^{m} \bar{\varepsilon}_{\mu_j} + \sum_{j=1}^{m'} \bar{\varepsilon}_{\nu_j}.$$

Let $a, b, c, d \in P$ satisfy

$$d = a + \sum_{j=1}^{m} \bar{\varepsilon}_{\kappa_j} \qquad c = d + \sum_{j=1}^{m'} \bar{\varepsilon}_{\lambda_j} \qquad b = a + \sum_{j=1}^{m'} \bar{\varepsilon}_{\nu_j} \qquad c = b + \sum_{j=1}^{m} \bar{\varepsilon}_{\mu_j}.$$

Then the $m \times m'$-fold fusion of W is defined as the antisymmetrized product of the m'-fold fusion of W in the horizontal direction:

$$W^{(m,m')}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| v\right) = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \prod_{j=1}^{m} W^{(1,m')}_{II}\left(\begin{matrix} b^\sigma_j & a_j \\ b^\sigma_{j-1} & a_{j-1} \end{matrix}\middle| z - \frac{m+1}{2} + j\right) \tag{A.3}$$

where

$$a_0 = a \qquad a_j = a_{j-1} + \bar{\varepsilon}_{\kappa_j}\ (1 \leqslant j \leqslant m) \qquad a_m = d$$

and

$$b_0^\sigma = b \qquad b_j^\sigma = b_{j-1}^\sigma + \bar{\varepsilon}_{\mu_{\sigma(j)}} \, (1 \leqslant j \leqslant m) \qquad b_m^\sigma = c.$$

The $W^{(m,m')}$ can also be defiend as the antisymmetrized product of the m-fold fusion of W in the vertical direction:

$$W^{(m,m')}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| z\right) = \sum_{\sigma\in\mathfrak{S}_{m'}} \operatorname{sgn}\sigma \prod_{j=1}^{m'} W^{(m,1)}\left(\begin{matrix} d_j & d_{j-1} \\ a_j^\sigma & a_{j-1}^\sigma \end{matrix}\middle| z + \frac{m'+1}{2} - j\right) \tag{A.4}$$

where

$$d_0 = d \qquad d_j = d_{j-1} + \bar{\varepsilon}_{\lambda_j} \, (1 \leqslant j \leqslant m) \qquad d_{m'} = c$$

and

$$a_0^\sigma = a \qquad a_j^\sigma = a_{j-1}^\sigma + \bar{\varepsilon}_{\nu_{\sigma(j)}} \, (1 \leqslant j \leqslant m') \qquad a_{m'}^\sigma = b.$$

The explicit expressions of fused Boltzmann weights in both the horizontal and vertical directions are given as follows [22]:

$$W^{(1,m)}\left(\begin{matrix} a+\bar{\varepsilon}_\lambda+\bar{\varepsilon}_\Lambda & a+\bar{\varepsilon}_\lambda \\ a+\bar{\varepsilon}_\Lambda & a \end{matrix}\middle| z\right) = (-1)^{m-1} r_m(z) \frac{[z+\frac{m-1}{2}]}{[z+\frac{m+1}{2}]} \prod_{j=1}^{m} \frac{[a_{\lambda\lambda_j}-1]}{[a_{\lambda\lambda_j}]}$$

$$W^{(1,m)}\left(\begin{matrix} a+\bar{\varepsilon}_\lambda+\bar{\varepsilon}_\Lambda & a+\bar{\varepsilon}_{\lambda_1} \\ a+\bar{\varepsilon}_\Lambda & a \end{matrix}\middle| z\right) = (-1)^{m-1} r_m(z) \frac{[z+\frac{m-1}{2}+a_{\lambda\lambda_1}][1]}{[z+\frac{m+1}{2}][a_{\lambda\lambda_1}]} \prod_{j=2}^{m} \frac{[a_{\lambda\lambda_j}-1]}{[a_{\lambda\lambda_j}]}$$

$$W^{(1,m)}\left(\begin{matrix} a+\bar{\varepsilon}_{\lambda_1}+\bar{\varepsilon}_\Lambda & a+\bar{\varepsilon}_{\lambda_1} \\ a+\bar{\varepsilon}_\Lambda & a \end{matrix}\middle| z\right) = (-1)^{m-1} r_m(z) \prod_{j=2}^{m} \frac{[a_{\lambda_1\lambda_j}]}{[a_{\lambda_1\lambda_j}+1]}$$

$$W^{(m,1)}\left(\begin{matrix} a+\bar{\varepsilon}_\lambda+\bar{\varepsilon}_\Lambda & a+\bar{\varepsilon}_\Lambda \\ a+\bar{\varepsilon}_\lambda & a \end{matrix}\middle| z\right) = (-1)^{m-1} r_m(z) \frac{[z+\frac{m-1}{2}]}{[z+\frac{m+1}{2}]} \prod_{j=1}^{m} \frac{[a_{\lambda_j\lambda}-1]}{[a_{\lambda_j\lambda}]} \tag{A.5}$$

$$W^{(m,1)}\left(\begin{matrix} a+\bar{\varepsilon}_\lambda+\bar{\varepsilon}_\Lambda & a+\bar{\varepsilon}_\Lambda \\ a+\bar{\varepsilon}_{\lambda_m} & a \end{matrix}\middle| z\right) = (-1)^{m-1} r_m(z) \frac{[z+\frac{m-1}{2}+a_{\lambda\lambda_m}][1]}{[z+\frac{m+1}{2}][a_{\lambda\lambda_m}]} \prod_{j=1}^{m-1} \frac{[a_{\lambda_j\lambda_m}-1]}{[a_{\lambda_j\lambda_m}]}$$

$$W^{(m,1)}\left(\begin{matrix} a+\bar{\varepsilon}_{\lambda_m}+\bar{\varepsilon}_\Lambda & a+\bar{\varepsilon}_\Lambda \\ a+\bar{\varepsilon}_{\lambda_m} & a \end{matrix}\middle| z\right) = (-1)^{m-1} r_m(z) \prod_{j=1}^{m-1} \frac{[a_{\lambda_j\lambda_m}-1]}{[a_{\lambda_j\lambda_m}]}.$$

Here we denote $\bar{\varepsilon}_{\lambda_1} + \cdots + \bar{\varepsilon}_{\lambda_m}$ by $\bar{\varepsilon}_\Lambda$ for simplicity.

When $m = n-1$ we identify W^{Ω,Ω^*} (resp. $W^{\Omega^*,\Omega}$) with $W^{(1,n-1)}$ (resp. $W^{(n-1,1)}$) as follows:

$$W^{\Omega_{z_1},\Omega^*_{z_2}}\begin{pmatrix} c & d \\ b & a \end{pmatrix} = (-1)^{n-1+\lambda+\nu} W^{(1,n-1)}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| z_1 - z_2\right) \tag{A.6}$$

where $b - a = -\bar{\varepsilon}_\nu$, $c - d = -\bar{\varepsilon}_\lambda$, and (b, c), (a, d) are admissible, and

$$W^{\Omega^*_{z_1},\Omega_{z_2}}\begin{pmatrix} c & d \\ b & a \end{pmatrix} = (-1)^{n-1+\mu+\kappa} W^{(n-1,1)}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| z_1 - z_2\right) \tag{A.7}$$

where $d - a = -\bar{\varepsilon}_\kappa$, $c - b = -\bar{\varepsilon}_\mu$, and (a, b), (d, c) are admissible.

The crossing symmetries are as follows:

$$W^{\Omega_{z_1},\Omega^*_{z_2}}\begin{pmatrix} c & d \\ b & a \end{pmatrix} = \frac{G_b}{G_c} W\left(\begin{matrix} d & a \\ c & b \end{matrix}\middle| z_2 - z_1 - \frac{n}{2}\right) \tag{A.8}$$

and

$$W^{\Omega^*_{z_1},\Omega_{z_2}}\begin{pmatrix} c & d \\ b & a \end{pmatrix} = \frac{G_b}{G_a} W\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| z_2 - z_1 - \frac{n}{2}\right). \tag{A.9}$$

Appendix B. Proof of the reflection equation

The aim of this appendix is to give a simple sketch of the proof of the claim that (2.14) solves the reflection equation (2.13). Since the K-matrix is diagonal in the sense that $K(a, b, c|z) = 0$ unless $b = c$, it is sufficient to consider the case $a = e = g$ in (2.13). In this case the reflection equation reduces to

$$\sum_d K\left(d\,\begin{matrix}a\\a\end{matrix}\middle|z_1\right) K\left(f\,\begin{matrix}a\\a\end{matrix}\middle|z_2\right) W\left(\begin{matrix}c & f\\ d & a\end{matrix}\middle|z_1+z_2\right) W\left(\begin{matrix}c & d\\ b & a\end{matrix}\middle|z_1-z_2\right)$$
$$= \sum_d K\left(d\,\begin{matrix}a\\a\end{matrix}\middle|z_1\right) K\left(b\,\begin{matrix}a\\a\end{matrix}\middle|z_2\right) W\left(\begin{matrix}c & f\\ d & a\end{matrix}\middle|z_1-z_2\right) W\left(\begin{matrix}c & d\\ b & a\end{matrix}\middle|z_1+z_2\right). \tag{B.1}$$

Note that (B.1) holds as $0 = 0$ unless the quartet (a, b, c, f) is admissible. Assume that (a, b, c, f) is admissible. Then there are the following three cases:

(i) $b = f = a + \bar{\varepsilon}_\mu$, $c = a + 2\bar{\varepsilon}_\mu$;
(ii) $b = f = a + \bar{\varepsilon}_\mu$, $c = a + \bar{\varepsilon}_\mu + \bar{\varepsilon}_\nu$ $(\mu \neq \nu)$;
(iii) $b = a + \bar{\varepsilon}_\mu$, $f = a + \bar{\varepsilon}_\nu$, $c = a + \bar{\varepsilon}_\mu + \bar{\varepsilon}_\nu$ $(\mu \neq \nu)$.

For case (i) equation (B.1) is trivial because the only non-zero terms of both sides result from $d = a + \bar{\varepsilon}_\mu$. It is also easy to prove case (ii). Up to now we did not use the explicit form of the K-matrix (2.14) except for its diagonal property.

Let us prove case (iii). By substituting (2.8) and (2.14) into (B.1), the reflection equation (2.13) is equivalent to

$$\frac{[\bar{a}_\nu + \eta + z_2]}{[\bar{a}_\nu + \eta - z_2]}\left(\frac{[\bar{a}_\mu + \eta + z_1]}{[\bar{a}_\mu + \eta - z_1]}[z_1 + z_2][a_{\mu\nu} - z_1 + z_2]\right.$$
$$\left. + \frac{[\bar{a}_\nu + \eta + z_1]}{[\bar{a}_\nu + \eta - z_1]}[z_1 - z_2][a_{\mu\nu} + z_1 + z_2]\right)$$
$$= \frac{[\bar{a}_\mu + \eta + z_2]}{[\bar{a}_\mu + \eta - z_2]}\left(\frac{[\bar{a}_\mu + \eta + z_1]}{[\bar{a}_\mu + \eta - z_1]}[z_1 - z_2][a_{\mu\nu} - z_1 - z_2]\right.$$
$$\left. + \frac{[\bar{a}_\nu + \eta + z_1]}{[\bar{a}_\nu + \eta - z_1]}[z_1 + z_2][a_{\mu\nu} + z_1 - z_2]\right).$$

Let F stand for the difference of both sides as the function of z_1. It is easy to show that the poles at $z_1 = \bar{a}_\mu + \eta$ and $z_2 = \bar{a}_\nu + \eta$ are spurious. Thus the function F is an entire function of z_1 with the quasi double periodicities. Suppose that F is not identically zero. Then the transformation properties of F contradict the positions of the zeros at $z_1 = \pm z_2$. We therefore obtain $F = 0$ and conclude that (2.14) solves the reflection equation (2.13).

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